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Unconditional basis of Banach spaces and Weak*-sequential property

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Abstract

The main purpose of this article is to answer the Plichko's question; let X be any Banach space and $\Gamma \subset X^*$ be any total set if $Ba(X;\ \sigma(X;\ \Gamma))\ = Ba(X; weak)$; then does imply every weakly* sequentially closed sub-space of X^* is weakly* closed? Also, answer the question: Let $A\ \subseteq X^*$ such that A is convex and bounded and for each element $a\ \in\ \bar{A}^{weak^*}$ then $a\ \in\ \bar{B}^{weak^*}$for a countable subset B of A; Then does it imply $(X;\ \sigma(X;\ \Gamma))\ = Ba(X; weak)$?

1. Introduction

Let X^* be the dual space of the Banach space X and B_{X^*} be a unit ball in X^*. Then:

1. X ([2]) is said to have weak*-angelic dual if $A\ \subseteq\ X^*$; A is closed and bounded subset of X^*, for all $f\ \in\ A$ there exist a sequence $(f_n)_{n\in\mathbb{N}}\ \in\ A$ such that $(f_n)\ \to f$;

2. X ([3; page.352]) has property $(\mathcal{E})$ (Efremov's property) if for every $\subseteq\ X^*$; A is bounded Convex, and for all $f \in \bar{A}^{weak^*}$ where $\bar{A}^{weak^*}$ is a weak* closure of A, then there exist a sequence$(f_n)_{n\in\mathbb{N}}\ \in\ A$ such that (f_n) $\xrightarrow{weak^*} f$;

3. X ([10]) has property $(\mathcal{E}')$, if for every $A \subseteq B_{X^*}$, A is weak*-sequentially closed convex, such that A is weak*- closed;

4. X (see[2]) has property $(\mathcal{D}')$ if $A \subseteq\ X^*$; A is a closed subspace. $\forall\ \{f_\lambda\}_{\lambda\in I} \in A\ , if\ \ f_\lambda \to f\ then\ f \in A$.

5. If Γ is a total subset of X^* (for a vector space X a set $S\ \subseteq\ X^*$ is a total set if every $f(x) =\ 0\ \ \forall f\ \ \in S$; then $x\ =\ 0$) then the σ-algebra generated by Γ; $Ba\big(X, \sigma(X, \Gamma)\big)$ is a subset of $Ba(X, weak)$.
6. X (see. [4]) has *Gulisashvili's property*$(\mathcal{D})$ if $Ba\big(X, \sigma(X, \Gamma)\big) = Ba(X, weak)$ for each total set $\Gamma \sqsubseteq X^*$;

7. Let $A \subseteq\ X^*$ be a bounded and closed set.. Then X [1] is said to have *Corson's property* $(\mathcal{C})$ if for every element $a \in \bar{A}^{weak^*}$ then $a \in \bar{B}^{weak^*}$ for a countable subset B of A.

Some relationships between these properties mentioned above have been stated. Plichko in [2] shows that there is a relation between some of these properties as follows:
Weak*-angelic dual space implies property $(\mathcal{E})$ implies property $(\mathcal{E}')$ implies property $(\mathcal{D}')$ implies property $(\mathcal{D})$.

Also, Plichko in [2] proved that the space (JL_2), Johnson-Lindenstrauss, has property ($\mathcal{D}$) and ($\mathcal{D}'$), but does not have weak*-angelic dual. Avilés et al. [12] give us a new diagram:

weak-angelic* dual space implies property ($\mathcal{E}$) implies property ($\mathcal{E}'$) implies property ($\mathcal{C}$).

The Banach space [1] $JL_2(\mathcal{F})$; Johnson-Lindenstrauss space, is the completion of the $span(c_0 \cup \{ \underline{X}^{\mathcal{K}_j} : j \in \Gamma) \subseteq \ell_\infty$ with respect to the norm:

$$\left\|x+\sum_{j=1}^{n} a_j\,\chi\mathcal{K}_j\right\|_{JL_2(\mathcal{F})} = \left\{\left\|x+\sum_{j=1}^{n} a_j\,\chi\mathcal{K}_j\right\|_\infty , \left(\sum_{j=1}^{n} a_j\,\chi\mathcal{K}_j\right)^{\frac{1}{2}}\right\}$$

where $\mathcal{F} = \{\mathcal{K}_j : j \in \Gamma\}$,$x \in c_0$ and $a_1; \dots ;$ $a_j \in \mathbb{R}$, $\chi\mathcal{K}$is the characteristic function of a set $\mathcal{K} \subseteq \mathbb{N}$ and the supremum norm on ℓ_∞ **.**

$JL_2(\mathcal{F}+)$ [14] is a Banach space that every bounded $(x_k^*)_{k\in\mathbb{N}}$ for which zero is weak*-cluster point has a subsequence $(x_{kn}^*)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \frac{1}{n}(x_{k_1}^*, \dots, x_{k_n}^*) = 0$.

$JL_2(\mathcal{F}-)$ [14] is a Banach space such that $\mathcal{F}-$ is constructed in such a way that $convex(e_k^* ; k \in \mathbb{N})$ does not contain weak*-null sequence in $JL_2(\mathcal{F}-)$.

Edgar in [5] shown that JL_2 has property ($\mathcal{C}$). As well Avilés [1] proved that $JL_2(\mathcal{F}+)$ has property ($\mathcal{E}$); but $JL_2(\mathcal{F}-)$ do not have property ($\mathcal{E}$) for all $\mathcal{F}+$ and $\mathcal{F}-$ are two maximal almost disjoint (MAD) families. Gonzalo in ([10, Theorem 9] answers negatively of the Plichko's question asked in [2] that JL_2, has weak*-sequential dual ball (which implies that JL_2 has property ($\mathcal{E}'$)) but it has not weak*-angelic dual.

There is a diagram to represent all possible relations between those above properties as in [1].

$$\begin{tikzcd}
\boxed{\text{weak*-angelic dual}} \arrow[r, Rightarrow] & \boxed{(\mathcal{E})} \arrow[r, Rightarrow] & \boxed{(\mathcal{E}')} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \boxed{(\mathcal{D}')} \arrow[r, Rightarrow] & \boxed{(\mathcal{D})} \\
 & & \boxed{(\mathcal{C})} & &
\end{tikzcd}$$

Many researchers have asked about the converse of the above diagram. Plichko in [2] demanded:

1. Does property ($\mathcal{D}$) imply property ($\mathcal{D}'$)?
2. Does property ($\mathcal{D}'$) imply property ($\mathcal{E}$)?

And Plichko and Yost in [3] asked: Does property ($\mathcal{C}$) of Corson imply property ($\mathcal{E}$)? Also, Plichko's question in [14]: a Banach space X has property ($\mathcal{E}$) if it has property ($\mathcal{E}'$)? In general, the converse of the above diagram for any Banach space is not correct. But, in any finite dimensional Banach spaces, we have the converse, because every Banach spaces of finite-dimension are separable spaces. But the separable space has weak*-angelic dual [5, page 565]. So all these properties mentioned above are equivalents in the finite-dimensional Banach spaces.

In this article, we will put a condition, unconditional basis; on Banach spaces of infinite dimension to answer the following questions and all questions asked above. Under what conditions the Banach space X has

Q1; Property ($\mathcal{D}$) $\Longrightarrow$ X has property ($\mathcal{D}'$)? [4]

Q2; Property ($\mathcal{D}$) $\Longrightarrow$ X has weak*- angelic dual?

Q3; Property ($\mathcal{C}$) $\Longrightarrow$ X has weak*- angelic dual?

Q4; Property ($\mathcal{D}$) $\Longrightarrow$ X has property ($\mathcal{C}$)?

Q5; Property ($\mathcal{C}$) $\Longrightarrow$ X has property ($\mathcal{D}$)?

If we can show that (Q 2), then we will answer (Q 1) and (Q 4) questions. Also, if we can show that (Q 3), then we will answer (Q 5).

2. Preliminary

In this section, we supposed that X is a Banach Space.

Definition 2.1. [6] A collection $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma} \subset X \times X^*$ is a biorthogonal system in X such that $f_\alpha(x_\gamma) = \begin{cases} 1 & if\ \gamma = \alpha \\ 0 & if\ \gamma \neq \alpha \end{cases}$.

Definition 2.2. [6, Page 1759] A biorthogonal system $\{x_\alpha, f_\alpha\}_{\alpha\in\Gamma}$ is unconditional basis for X if for each $x \in X$; $x = \sum f_\alpha(x)x_\alpha$; there is a finite set $A \subset \Gamma$ for a given $\epsilon > 0$ such that $\|x - \sum_{\alpha\in B} f_\alpha(x)x_\alpha\| \leq \epsilon$, whenever $A \subset B \subset \Gamma$ and B is a finite set.

Definition 2.3 [6] A biorthogonal system $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ in X is M-basis (Markushevich basis) if $\overline{span}\{x_\gamma\}_{\gamma\in\Gamma} = X$ and $\overline{span}^{weak^*}\{x^*_\gamma\}_{\gamma\in\Gamma} = X^*$.

Definition 2.4 [6] If $X = \overline{span}K$ for a subset K of X, then X is said to be weakly compactly generated (shortly, WCG) .

Definition 2.5. [6] If unit dual ball of X is Corson compact in its weak*- topology, then X is called weakly Lendelöf determined (WLD).

Theorem 2.1. [7, Theorem 3.4.7] X with unconditional basis either is reflexive or contains a copy of c_0 , or contains a copy of ℓ_1.

Theorem 2.2. [8, Theorem 7.43] A Banach space X with an unconditional basis $\{x_\gamma; f_\gamma\}_{\gamma\in\Gamma}$ is weakly Lindelöf Determined (WLD) if and only if there is no isomorphic copy of $\ell_1(\omega_1)$ in X where ω_1 is a first uncountable ordinal.

Theorem 2.3. [6, Page 1765] X admits Markushevich basis if it has an unconditional basis.

Theorem 2.4. [6, Theorem 4.4] For a X which has Markushevich basis is WLD if and only if it has property $(\mathcal{C})$.

Theorem 2.5. [9, Proposition 1.2] Weakly Lendelöf determined space X is equivalent to $(B_{X^*}; weak^*)$ is Corson-compact space.

Theorem 2.6. [8, Proposition 5.27] If X is Corson-compact space, then it is angelic.

Theorem 2.7. [6, Page 1757] ℓ_∞ does not contain a subspace isomorphic to the space JL_2.

Theorem 2.8. [12] If X has property $(\mathcal{D})$, then X contains no subspace isomorphic to $\ell_1(\omega_1)$.

Theorem 2.9. [1] For X has Weak*- angelic dual implies X has property $(\mathcal{E})$ implies property $(\mathcal{E}')$ implies property $(\mathcal{C})$.

Theorem 2.10. [1] For X has Weak*- angelic dual implies X has property $(\mathcal{E})$ implies property $(\mathcal{E}')$ implies property $(\mathcal{D}')$ implies property $(\mathcal{D})$.

Theorem 2.11. [5, Page 565] Every reflexive Banach space X has weak*-angelic dual.

3 Main Results

Johnson-Lindenstrauss JL_2 space has many properties stated in section 1, such as property $(\mathcal{C})$, property $(\mathcal{E})$. But $JL_2(\mathcal{F}-)$ does not hqve property $(\mathcal{E})$. And it has property $(\mathcal{E}')$. Now according to our condition we see that the space of Johnson-Lindenstrauss JL_2 has no unconditional basis.

Theorem 3.1 The space of Johnson-Lindenstrauss JL_2 has no unconditional basis (x_γ).

Proof: Suppose that the space Johnson-Lindenstrauss JL_2 has an unconditional basis. Then by Theorem (2.1) either

1. JL_2 is reflexive. But Theorem 2.11 implies that JL_2 has weak*-angelic dual. This is contradiction.

2. Or JL_2 contains a copy of c_0. But c_0 is a subspace of ℓ_∞which is a contradiction to Theorem (2.7).
3. Or JL_2 contains a copy of ℓ_1, but ℓ_1 is a subspace of ℓ_∞, therefore JL_2 contains a copy of subspace of ℓ_∞ . Which is a contradiction because of Theorem (2.7). Hence JL_2 does not have an unconditional basis (x_γ).

Theorem 3.2. Let $\{x_\gamma; f_\gamma\}_{\gamma\in\Gamma}$ be an unconditional basis for *X*. Then, if *X* has property ($\mathcal{D}$), then *X* has weak*-angelic dual.

Proof: Let *X* has property ($\mathcal{D}$). Then by Theorem (2.8) there is no subspace isomorphic to $\ell_1(\omega_1)$ in *X*, then *X* is WLD (Theorem (2.2)). Therefore $(B_{X^*}; weak^*)$ is Corson compact space Theorem (2.5), but from Theorem (2.6) the space is angelic if it is Corson compact, then the space $(B_{X^*}; weak^*)$ is angelic. Hence *X* has weak*- angelic dual.

Corollary 3.1 Let $\{x_\gamma; f_\gamma\}_{\gamma\in\Gamma}$ be an unconditional basis for *X*. Then, *X* has weak*-angelic dual if and only if *X* has property ($\mathcal{D}$).

Theorem 3.3 Let $\{x_\gamma; f_\gamma\}_{\gamma\in\Gamma}$ be an unconditional basis for *X*. Then, if *X* has property ($\mathcal{C}$), then *X* has weak*-angelic dual.

Proof: Let X has property ($\mathcal{C}$). From Theorem (2.3) *X* admits a Markushevich basis. Then *X* is WLD by Theorem (2.4). Therefore $(B_{X^*}; weak^*)$ is Corson-compact space by Theorem (2.5), but from Theorem (2.6) the space $(B_{X^*}; weak^*)$ is angelic. Hence *X* has weak*-angelic dual.

Corollary 3.2 Let $\{x_\gamma; f_\gamma\}_{\gamma\in\Gamma}$ be an unconditional basis for *X*. Then, the property ($\mathcal{D}$) is equivalent to the property ($\mathcal{C}$).

Proof:

1. Let X has property ($\mathcal{D}$), then by Theorem (2.8) X contains no subspace isomorphic to $\ell_1(\omega_1)$ Therefore by Theorem (2.2) *X* is weakly Lindelöf Determined (WLD) space; but *X* is a Banach space with an unconditional basis so that *X* admits M-basis by Theorem (2.3). Hence from Theorem (2.4) *X* has property ($\mathcal{C}$).
2. Let *X* has property ($\mathcal{C}$). Then from Theorem (3.3) and Theorem (2.10) *X* has property ($\mathcal{D}$).

The following corollaries will be induced by Theorem (2.9), (2.10), (3.2), and Theorem (3.3).

Corollary 3.3 Let $\{x_\gamma; f_\gamma\}_{\gamma\in\Gamma}$ be an unconditional basis for *X*. Then:

1. *X* has weak*- angelic dual if and only if *X* has property ($\mathcal{E}$).
2. *X* has property ($\mathcal{E}$) if and only if *X* has property ($\mathcal{C}$).
3. *X* has property ($\mathcal{E}$) if and only if *X* has property ($\mathcal{D}$).
4. *X* has weak*-angelic dual if and only if *X* has property ($\mathcal{E}'$).
5. *X* has property ($\mathcal{E}$) if and only if *X* has property ($\mathcal{E}'$).
6. *X* has property ($\mathcal{D}$) if and only if *X* has property ($\mathcal{D}'$).

Our conclusion from all theorems and corollaries are answering proposed all questions asked in section.1.

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